Pergamon

PII: S0021-8928(96)00060-3

J. Appl. Maths Mechs, Vol. 60, No. 3, pp. 473–486, 1996

0021–8928/96 $24.00+0.00

THE UNSTEADY ESCAPE OF A GAS THROUGH A PLANE SLIT INTO A VACUUM†

Ya. M. KAZHDAN

Moscow

(*Received* 16 *January* 1995)

The escape of a gas through a plane slit into a vacuum is considered and the flow in the region where it can be represented by a self-similar solution of the corresponding gas-dynamic system is considered. The asymptotic forms in the neighbourhood of singular flow surfaces are investigated. In particular, the deviation of the free surface from its analogue in the Prandtl–Mayer type solution is determined. Copyright © 1996 Elsevier Science Ltd.

The unsteady escape of a gas into a vacuum has been considered in a number of publications (for example, [1–4]).

1. Consider the following situation. At the instant $t = 0$ a gas begins to escape into a vacuum through a plane slit $-a < x < a, y \leqslant 0$ filled with an ideal gas having an adiabatic index γ. When $t = 0$ the gas is at rest and the velocity of sound $c = c_0$. The flow that occurs when $t \geqslant 0$ is symmetrical about the plane $x = 0$, and up to the instant $t = t_0^*$ two regions can be distinguished in xyt space: (1) a region of unperturbed one-dimensional flow, corresponding to a plane Riemann wave, and (2) a region of perturbed two-dimensional flow.

The asymptotic forms of the flow at times close to the instant when expansion begins, and, at finite times, in the neighbourhood of the boundary between the Riemann wave and the boundary of the expansion, can be found by the method of matched asymptotic expansions [5–7]. In view of the symmetry of the flow we will consider only the region $(x \geqslant 0, y, t)$.

For our further calculations it will be more convenient to use the following dimensionless variables

$$x = a\bar{x}, \quad y = a\bar{y}, \quad t = t_0\bar{t}, \quad v_x = c_0\bar{v}_x, \quad v_y = c_0\bar{v}_y, \quad c = c_0\bar{c}, \quad t_0 = a / c_0 \tag{1.1}$$

(v_x and v_y are the corresponding components of the velocity). In the new variables, the slit is defined by the inequalities $-1 \leqslant x \leqslant 1$, and the velocity of sound in the gas up to the time when the gas begins to escape is $c_0 = 1$. The bar on the symbols will henceforth be omitted. The flow is self-similar in the regions considered up to a certain time $t = t_0^*$. The boundary between regions 1 and 2 coincides with the characteristic emerging from the boundary of the slit $x = 1, y = 0$ at the instant $t = 0$. The instant t_0^* coincides with the instant when this characteristic meets the plane $x = 0$.

The self-similar variables and functions will be as follows:

$$\xi = \frac{x-1}{t}, \quad \eta = \frac{y}{t}, \quad v_x = v_x(\xi,\eta), \quad v_y = v_y(\xi,\eta), \quad c = c(\xi,\eta) \tag{1.2}$$

In self-similar variables the flow is governed by the equations

$$(v_x - \xi)\frac{\partial v_x}{\partial \xi} + (v_y - \eta)\frac{\partial v_x}{\partial \eta} + \frac{h-1}{2}\frac{\partial f}{\partial \xi} = 0$$

$$(v_x - \xi)\frac{\partial v_y}{\partial \xi} + (v_y - \eta)\frac{\partial v_y}{\partial \eta} + \frac{h-1}{2}\frac{\partial f}{\partial \eta} = 0 \tag{1.3}$$

$$(v_x - \xi)\frac{\partial f}{\partial \xi} + (v_y - \eta)\frac{\partial f}{\partial \eta} + \frac{2f}{h-1}\left(\frac{\partial v_x}{\partial \xi} + \frac{\partial v_y}{\partial \eta}\right) = 0 \quad \left(f = c^2, \quad h = \frac{\gamma+1}{\gamma-1}\right)$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 3, pp. 475–488, 1996.

In the self-similar ξ, η plane it will be convenient henceforth to use polar coordinates φ and r, where φ is the angle which the ray from the point $\xi = 0$, $\eta = 0$ makes with the semiaxis ($\eta = 0$, $\xi \leqslant 0$), and r is the distance from the origin ($\xi = 0$, $\eta = 0$) to the point ξ, η ($r^2 = \xi^2 + \eta^2$), and the corresponding components of the velocity $v_r = \partial r/\partial t$, $v_\varphi = -r\partial\varphi/\partial t$. Here

$$\xi = -r\cos\varphi, \quad \eta = r\sin\varphi \tag{1.4}$$
$$v_x = -v_r\cos\varphi - v_\varphi\sin\varphi, \quad v_y = v_r\sin\varphi - v_\varphi\cos\varphi$$

In polar coordinates the gas-dynamic system can be rewritten as follows:

$$(v_r - r)\frac{\partial v_r}{\partial r} - \frac{v_\varphi}{r}\frac{\partial v_r}{\partial\varphi} - \frac{v_\varphi^2}{r} + \frac{h-1}{2}\frac{\partial f}{\partial r} = 0$$
$$(v_r - r)\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\rho v_\varphi}{r} - \frac{h-1}{2r}\frac{\partial f}{\partial\varphi} = 0 \tag{1.5}$$
$$(v_r - r)\frac{\partial f}{\partial r} - \frac{v_\varphi}{r}\frac{\partial f}{\partial\varphi} + \frac{2}{h-1}f\left[\frac{\partial v_r}{\partial r} - \frac{1}{r}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_r}{r}\right] = 0$$

2. The flow in the first region—the Riemann wave—is defined by the formulae

$$v_x = 0, \quad v_y = \frac{h-1}{h}(\eta+1), \quad c = \frac{1}{h}(h-1-\eta) \tag{2.1}$$

The equation of the characteristic, which separates the Riemann wave and the perturbed two-dimensional flow, is found from system (1.3) and the values of the functions in the Riemann wave

$$\frac{d\eta}{d\xi} = -\frac{2c\xi}{\xi^2 - c^2} \tag{2.2}$$

With the initial values $\xi = 0$, $\eta = 0$ the solution of Eq. (2.2), i.e. the equation of the characteristic, will be

$$\xi = -[h(h-2)]^{-1/2}\left[1 - \left(1 - \frac{\eta}{h-1}\right)^{h-2}\right]^{1/2}(h-1-\eta) \tag{2.3}$$

The Riemann wave will correspond to the flow in the first region up to the instant t_0^* when the separating characteristic meets the plane $x = 0$. The lines of intersection in the ξ, η plane will correspond to the point where the function $\xi(\eta)$, which defines this characteristic, is a minimum

$$\xi_{\min} = -\frac{h-1}{h}\left(\frac{2}{h}\right)^{1/(h-2)}, \quad \eta_{\min} = (h-1)\left[1 - \left(-\frac{2}{h}\right)^{1/(h-2)}\right] < h-1 \tag{2.4}$$

In the $x = 0$ plane we have $\xi = -1/t$. Consequently

$$t_0^* = \frac{h}{h-1}\left(\frac{h}{2}\right)^{1/(h-2)} \tag{2.5}$$

Hence, when $t = t_0^*$ the first region only exists in the range $0 < \eta < \eta_{\min}$ and, consequently, the free boundary, the equation of which in the Riemann wave is $\eta = h - 1$, does not occur in the first region.

In polar coordinates the Riemann wave in the first region is defined by the formulae

$$v_r = \frac{h-1}{h}(1 + r\sin\varphi)\sin\varphi, \quad v_\varphi = -\frac{h-1}{h}(1 + r\sin\varphi)\cos\varphi \tag{2.6}$$

$$f=\left(\frac{h-1}{h}-\frac{r\sin\varphi}{h}\right)^2$$

while the equation which defines the separating characteristic has the form

$$\frac{dr}{d\varphi}=r\frac{v_\varphi(v_r-r)-\sqrt{f}\,r\cos\varphi}{f-v_\varphi^2} \tag{2.7}$$

The equation of the characteristic itself is given by the implicit function

$$r\cos\varphi=(h-1-r\sin\varphi)[h(h-2)]^{-1/2}\left[1-\left(1-\frac{r\sin\varphi}{h-1}\right)^{h-2}\right]^{1/2} \tag{2.8}$$

As $\varphi\to 0$, the equations of the characteristic and the functions of it have the following asymptotic forms

$$r\approx\frac{h-1}{h}\varphi+\frac{(h-1)(7h+6)}{12h^2}\varphi^3,\quad f\approx\left(\frac{h-1}{h}\right)^2\left[1-\frac{2}{h}\varphi^2+\frac{12-5h}{6h^2}\varphi^4\right]$$

$$v_r\approx-\frac{h-1}{h}\varphi\left[1+\frac{5h-6}{6h}\varphi^2+\frac{1}{4}\left(\frac{(h-1)(h-2)}{h^2}+\frac{1}{30}\right)\varphi^4\right] \tag{2.9}$$

$$v_\varphi\approx\frac{h-1}{h}\left[1+\frac{h-2}{2h}\varphi^2+\frac{1}{24}\frac{h^2+10h-12}{h^2}\varphi^4\right]$$

The following values correspond to the point where the characteristic meets the plane of symmetry $x=0$

$$\varphi=\varphi_0=\operatorname{arctg}\left(-\frac{\eta_{\min}}{\xi_{\min}}\right)=\operatorname{arctg}\left\{h\left[\left(\frac{h}{2}\right)^{1/(h-2)}-1\right]\right\}$$

$$r=(h-1)\left[1-2\left(\frac{2}{h}\right)^{1/(h-2)}+\left(\frac{1}{h^2}+1\right)\left(\frac{2}{h}\right)^{2/(h-2)}\right]^{1/2} \tag{2.10}$$

3. The method of determining the asymptotic form of gas-dynamic functions in the second region reduces to representing the gas-dynamic functions in the form of the first few terms of the asymptotic series and substituting them into system (1.5), after which one can obtain in an obvious way the equations which govern the coefficients of the expansion. Here the asymptotic forms on the boundaries of neighbouring regions obtained from the values of the functions in each of them must be identical in definite orders.

In the second region, the flow in the principal term is of the Prandtl–Mayer type, where all the gas-dynamic quantities are functions of the angle φ only. By (1.5) these functions are defined by the system of equations

$$\frac{v_\varphi}{r}(v_r'+v_\varphi)=0,\quad \frac{v_\varphi}{r}(v_r-v_\varphi')-\frac{h-1}{2r}f'=0$$

$$\frac{v_r}{r}f'+\frac{2}{h-1}f(v_\varphi'-v_r)=0 \tag{3.1}$$

and the initial data

$$\varphi = 0, \quad v_r = 0, \quad v_\varphi = -\frac{h-1}{h}, \quad f = \left(\frac{h-1}{h}\right)^2 \tag{3.2}$$

The solution of system (3.1) and (3.2) can be written in the explicit form

$$v_r = \frac{h-1}{\sqrt{h}}\sin\frac{\varphi}{\sqrt{h}}, \quad v_\varphi = -\frac{h-1}{h}\cos\frac{\varphi}{\sqrt{h}}, \quad f = \left(\frac{h-1}{h}\right)^2 \cos^2\frac{\varphi}{\sqrt{h}} \tag{3.3}$$

Here $0 \leqslant \varphi \leqslant \sqrt{(h)}\pi/2$. The line $\varphi = \sqrt{(h)}\pi/2$ corresponds to the free boundary when there are no perturbations. Obviously the proposed investigation only holds when $\sqrt{(h)}\pi/2 < \varphi_0$, where φ_0 is the angle corresponding to the half-plane which bounds the initial region of the vacuum. Since $\varphi_0 < 3\pi/2$, we will henceforth assume $h < 9$.

When $\varphi \to 0$ we obtain the asymptotic form

$$\begin{aligned} v_r &= \varphi\left(\frac{h-1}{h} - \frac{h-1}{6h^2}\varphi^2\right), \quad v_\varphi = -\frac{h-1}{h} + \frac{h-1}{2h^2}\varphi^2 \\ f &= \left(\frac{h-1}{h}\right)^2\left(1 - \frac{1}{h}\varphi^2\right) \end{aligned} \tag{3.4}$$

The asymptotic form of the flow when $\varphi \to 0$ along the separating characteristic (2.9) is identical with the corresponding asymptotic form of the Prandtl–Mayer solution (3.4) only in the principal term. Hence, to match the solutions, starting from the asymptotic forms (2.9) and (3.4) we will seek the asymptotic form of the solution in the part adjoining the characteristic of region 2, as $\varphi \to 0$ and $r \to 0$, in the form

$$\begin{aligned} v_r &= v_{r0}(\zeta)\varphi + v_{r1}(\zeta)\varphi^3 + v_{r2}(\zeta)\varphi^5 \\ v_\varphi &= v_{\varphi 0}(\zeta) + v_{\varphi 1}(\zeta)\varphi^2 + v_{\varphi 2}(\zeta)\varphi^4 \\ f &= f_0(\zeta) + f_1(\zeta)\varphi^2 + f_2(\zeta)\varphi^4 \quad (\zeta = r/\varphi) \end{aligned} \tag{3.5}$$

The boundary conditions for the equations defining the functions with subscripts 0 and 1, will be the values of these functions when $\zeta = \zeta_0 = (h-1)/h$, corresponding to the characteristic, and when $\zeta = \zeta_1 = 0$, corresponding to the Prandtl–Mayer solution. We have

$$\begin{aligned} &\zeta = \zeta_0: \quad v_{r0}(\zeta_0) = \frac{h-1}{h}, \quad v_{\varphi 0}(\zeta_0) = -\frac{h-1}{h}, \quad f_0 = \left(\frac{h-1}{h}\right)^2 \\ &v_{r1}(\zeta_0) = \frac{(h-1)(5h-6)}{6h^2}, \quad v_{\varphi 1}(\zeta_0) = -\frac{(h-1)(h-2)}{2h^2} \\ &f_1(\zeta_0) = -2\frac{(h-1)^2}{h^3} \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\zeta = \zeta_1: \quad v_{r0}(\zeta_1) = \frac{h-1}{h}, \quad v_{\varphi 0}(\zeta_1) = -\frac{h-1}{h}, \quad f_0 = \left(\frac{h-1}{h}\right)^2 \\ &v_{r1}(\zeta_1) = -\frac{h-1}{6h^2}, \quad v_{\varphi}(\zeta_1) = \frac{h-1}{2h^2} \\ &f_1(\zeta_1) = -\frac{(h-1)^2}{h^3} \end{aligned} \tag{3.7}$$

The functions with a subscript of zero are defined by the system of equations

$$\begin{aligned} &v_{r0}v'_{r0}\zeta+\frac{h-1}{2}f'_0\zeta-v_{\varphi 0}(v_{r0}+v_{\varphi 0})=0 \\ &v_{\varphi 0}v'_{\varphi 0}\zeta+\frac{h-1}{2}f'_0\zeta=0, \quad \frac{2}{h-1}f_0v'_{\varphi 0}+v_{\varphi 0}f'_0=0 \end{aligned} \tag{3.8}$$

Here and henceforth the prime denotes a derivative with respect to the corresponding variable.

The solution of the system consists of constants which are identical with the boundary conditions for the functions with a subscript of zero.

To obtain the equations which define the functions with a subscript of one, we will use a method which will be employed repeatedly later. If we substitute the first two terms of representation (3.5) into system (1.5), we obtain only two linearly independent equations defining the functions with a subscript of one

$$\begin{aligned} &\frac{1}{h}v'_{r1}\zeta-\frac{1}{2}f'_1\zeta-\frac{1}{h}(3v_{r1}+v_{\varphi 1})=0 \\ &\frac{2}{h}v'_{\varphi 1}\zeta-f'_1\zeta+2f_1-\frac{4}{h}v_{\varphi 1}+\frac{2(h-1)}{h^2}=0 \end{aligned} \tag{3.9}$$

Here the second equation can be replaced by the first integral, which, taking boundary conditions (3.6) and (3.7) into account, can be written in the form

$$v_{\varphi 1}-\frac{h}{2}f_1=\frac{h-1}{2h} \tag{3.10}$$

In order to obtain the lacking third equation, we will write two equations of the system which define the functions with a subscript of two

$$\begin{aligned} &v_{\varphi 0}v'_{\varphi 2}\zeta+\frac{h-1}{2}f'_2\zeta=4v_{\varphi 0}v_{\varphi 2}+2(h-1)f_2-(v_{r0}-\zeta)v'_{\varphi 1}\zeta+ \\ &+v_{\varphi 1}(2v_{\varphi 1}-2v'_{\varphi 1}\zeta)-v_{r0}v_{\varphi 1}-v_{\varphi 0}v_{r1} \\ &\frac{-2(h-1)}{h}v'_{\varphi 2}\zeta+\frac{h-1}{h}f'_2\zeta=-8\frac{h-1}{h^2}v_{\varphi 2}+4\frac{h-1}{h}f_2+ \\ &+\frac{2}{h-1}[f_0v'_{r1}\zeta+f_1v'_{\varphi 1}\zeta+(v_{\varphi 1}+v_{r0}-\zeta)f'_1\zeta+f_0v_{r1}-(v_{r0}-(h+1)v_{\varphi 1})f_1] \end{aligned}$$

Subtracting the first equation, multiplied by $2/h$, from the second, we obtain the lacking third equation.

Hence, using the first integral (3.10), the determination of the functions with a subscript of one can be reduced to integrating a system consisting of two differential equations

$$\begin{aligned} &\left(\frac{2h-3}{h}-2\zeta+\frac{2h}{h-1}v_{\varphi 1}\right)v'_{\varphi 1}\zeta+\frac{h-1}{h}v'_{r1}\zeta=\frac{h-1}{h}(1-2v_{\varphi 1})+\frac{h}{h-1}v^2_{\varphi 1} \\ &-v'_{\varphi 1}\zeta+v'_{r1}\zeta=v_{\varphi 1}+3v_{r1} \end{aligned} \tag{3.11}$$

Here, according to boundary conditions (3.6) and (3.7), it is required to obtain an integral curve connecting the points A and B, where

$$\begin{aligned} &A\colon\ \zeta=\frac{h-1}{h}, \quad v_{r1}=\frac{(h-1)(5h-6)}{6h^2}, \quad v_{\varphi 1}=-\frac{(h-1)(h-2)}{2h^2} \\ &B\colon\ \zeta=0, \quad v_{r1}=-\frac{h-1}{6h^2}, \quad v_{\varphi 1}=\frac{h-1}{2h^2} \end{aligned}$$

The exact solution of system (3.11) can be written explicitly as follows:

$$\upsilon_{r1} = -\left[\frac{(\alpha+\chi)^2(2\alpha\chi-5/2\alpha^2)}{3\alpha^2}+\frac{\alpha}{6h}\right],\quad \upsilon_{\varphi 1} = -\frac{1}{2}(\alpha+\chi)^2+\frac{\alpha}{2h}$$
$$\left(\alpha=\frac{h-1}{h},\quad \chi=\zeta-\frac{h-1}{h}\right) \tag{3.12}$$

From the first integral (3.10) we obtain the function

$$f_1 = -\frac{1}{h}(\alpha+\chi)^2+\frac{\alpha^2}{h^2} \tag{3.13}$$

which, as can be easily verified, satisfies the boundary conditions. Two conclusions, which are important for later investigations, follow from the formulae obtained.

1. In the asymptotic form obtained in powers of χ the coefficients of the zeroth and first powers of χ are identical with the corresponding coefficients of the unperturbed flow.

2. Since $\alpha + \chi = r/\varphi$, the solution obtained has the following form in powers of r

$$G=(\upsilon_r,\upsilon_\varphi,f)=G_0(\varphi)+r^2G_2(\varphi)+\ldots \tag{3.14}$$

The solution obtained can be extended in two directions: along the separating characteristics, and for finite values of φ, but for small values of r.

4. To obtain the asymptotic form of the solution in the neighbourhood of the characteristic, the equation of which we will denote by $r = r(\varphi)$, we will represent the gas-dynamic functions in the form of asymptotic series

$$G=G_0(\varphi)+(g-1)G_1(\varphi)+(g-1)^2G_2(\varphi)+\ldots,\quad g=r/r(\varphi) \tag{4.1}$$

Functions corresponding both to the unperturbed and the perturbed flows must satisfy the system of equations, which will be obtained for determining $G_1(\varphi)$. The initial asymptotic forms for the perturbed flow as $\varphi \to 0$ correspond to the functions determined in Section 3.

In view of conclusion 1 reached at the end of Section 3, the functions $G_0(\varphi)$ and $G_1(\varphi)$ will be identical with the corresponding solutions for the unperturbed flow. Substituting series (4.1) into system (1.5) we obtain a system of equations for determining $G_2(\varphi)$ in which only two equations are linearly independent. The lacking third equation is obtained by the method employed earlier.

We will write the system of equations for the functions with a subscript of three. The determinant of the system is also equal to zero here, but the right-hand sides contain functions with a subscript of two and their derivatives. The linear combination of the equations obtained with coefficients

$$1,\quad A=r'/r$$
$$B=[r\upsilon_{\varphi 0}\cos\varphi-\sqrt{f_0}(\upsilon_{r0}-r)]/[(\gamma-1)\sqrt{f_0}(f_0-\upsilon_{\varphi 0}^2)]$$

reduces the combination of the left-hand sides of the equations to zero, and gives the lacking third equation for determining the functions with a subscript of two. The following system of equations is thereby obtained

$$2S\upsilon_{r2}+(h-1)f_2=-\upsilon_{r1}(\upsilon_{r1}-r)+\left(\upsilon_{r0}'-\frac{r'}{r}\upsilon_{r1}\right)(\upsilon_{\varphi 1}-\upsilon_{\varphi 0})+$$
$$+\left(\upsilon_{r1}'-\upsilon_{r1}\frac{r'}{r}\right)\upsilon_{\varphi 1}+2\upsilon_{\varphi 0}\upsilon_{\varphi 1}-\upsilon_{\varphi 0}^2$$
$$2S\upsilon_{\varphi 2}+(h-1)\frac{r'}{r}f_2=-\upsilon_{\varphi 1}(\upsilon_{r1}-r)+\left(\upsilon_{r0}'-\frac{r'}{r}\upsilon_{\varphi 1}\right)(\upsilon_{\varphi 1}-\upsilon_{\varphi 0})+$$
$$+\left(\upsilon_{\varphi 1}-\frac{r'}{r}\upsilon_{\varphi 1}\right)-\upsilon_{\varphi 0}\upsilon_{r1}-\upsilon_{r0}\upsilon_{\varphi 1}+\upsilon_{r0}\upsilon_{\varphi 0}+\frac{h-1}{2}(f_1'-f_0')$$

$$\upsilon_{\varphi 0}\upsilon'_{r2}+\left(A\upsilon_{\varphi 0}+\frac{2}{h-1}Bf_0\right)\upsilon'_{\varphi 2}+\left(\frac{h-1}{2}A+\upsilon_{\varphi 0}B\right)f'_2=$$
$$=\left[2(\upsilon_{r1}-r)+\upsilon_{r1}+2(\upsilon_{\varphi 1}-\upsilon_{\varphi 0})+A(\upsilon_{\varphi 1}+\upsilon_{\varphi 0})+B\left(\frac{h+3}{h-1}f_1+\frac{2}{h-1}f_0\right)\right]\upsilon_{r2}+$$
$$+\left\{-\left(\upsilon'_{r0}-\upsilon_{r1}\frac{r'}{r}+2\upsilon_{r0}\right)+A\left[2(\upsilon_{r1}-r)-\upsilon'_{\varphi 0}+3\upsilon_{\varphi 1}\frac{r'}{r}+\upsilon_{\varphi 0}\right]-\right.$$
$$\left.-B\left(f'_0-\frac{h+3}{h-1}f_1\frac{r'}{r}\right)\right\}\upsilon_{\varphi 2}+ \qquad (4.2)$$
$$+B\left[2\left(\upsilon_{r1}-r+\upsilon_{\varphi 1}\frac{r'}{r}\right)+\frac{2}{h-1}\left(\upsilon_{r0}-\upsilon'_{\varphi 0}+\upsilon_{r1}+\upsilon_{\varphi 1}\frac{r'}{r}\right)\right]f_2-$$
$$-(\upsilon_{\varphi 1}-\upsilon_{\varphi 0})\,\upsilon'_{r1}-\upsilon_{\varphi 0}\upsilon_{r1}\frac{r'}{r}-\upsilon^2_{\varphi 1}+\upsilon_{\varphi 1}(2\upsilon_{\varphi 0}+\upsilon'_{r0})+$$
$$+A\left[\frac{h-1}{2}\left(f'_1-f_1\frac{r'}{r}\right)\upsilon_{\varphi 1}-\left(\upsilon'_{\varphi 1}-\upsilon_{\varphi 1}\frac{r'}{r}-\upsilon_{r1}\right)(\upsilon_{\varphi 1}-\upsilon_{\varphi 0})-\upsilon_{r0}\upsilon_{\varphi 1}\right]+$$
$$+B\left\{(f'_0-f'_1)\upsilon_{\varphi 0}+\upsilon_{\varphi 0}\left(f'_1-f_1\frac{r'}{r}\right)+\right.$$
$$\left.+\frac{2}{h-1}\left[-f_0\upsilon_{r1}+(\upsilon_{r1}-\upsilon_{r0}+\upsilon'_{\varphi 0}-\upsilon'_{\varphi 1})f_1+f_0(\upsilon'_{\varphi 1}-\upsilon_{\varphi 1})\frac{r'}{r}\right]\right\}$$

Here

$$S=\upsilon_{r0}-r+\frac{r'}{r}\upsilon_{\varphi 0},\quad \upsilon_{r0}=\frac{h-1}{h}(r\sin\varphi+1)\sin\varphi$$
$$\upsilon_{\varphi 0}=-\frac{h-1}{h}(r\sin\varphi+1)\cos\varphi,\quad f=\left(\frac{h-1}{h}-\frac{r\sin\varphi}{h}\right)^2 \qquad (4.3)$$
$$\upsilon_{r1}=\frac{h-1}{h}r\sin^2\varphi,\quad \upsilon_{\varphi 1}=-\frac{h-1}{h}r\sin\varphi\cos\varphi$$
$$f_1=-2\left(\frac{h-1}{h}-\frac{r\sin\varphi}{h}\right)r\sin\varphi$$

Substituting the expressions for $\upsilon_{r2}(\varphi)$ and $\upsilon_{\varphi 2}(\varphi)$, taken from the first two equations, into the third, we obtain a linear differential equation for the function $f_2(\varphi)$, which must be integrated in the limits $0<\varphi<\varphi_0=-\operatorname{arctg}\{h[1-(h/2)^{1/(h-2)}]\}<\pi/2$, where the coefficient of f'_2 is equal to $-2r\cos\varphi/[(\chi-1)\sqrt{f}]$ for the initial asymptotic form

$$\varphi\to 0;\quad f_2\approx-(h-1)^2h^{-3}\varphi^2 \qquad (4.4)$$

Singularities may arise in the solution only if some of the quantities $f_0-\upsilon^2_{\varphi 0}$, S vanish. However, it can be shown that these quantities vanish only when $\varphi=0$.

Hence, in the neighbourhood of the characteristic in the second region the functions with subscripts zero, one and two are continuous and have continuous derivatives. This assertion also holds for the coefficients for the asymptotic series with higher numbers, since the occurrence of a singularity in the corresponding equations is due to the fact that the expressions $f_0-\upsilon^2_{\varphi 0}$ and S vanish. Hence, we can assume characteristic (2.8) to be the boundary of the flow considered in the range $0<\varphi<\varphi_0$.

5. The asymptotic form of the flow as $r \to 0$ and finite φ, must be sought, starting from (3.14), in the form

$$\upsilon_r = \upsilon_{r0}(\varphi) + r^2\upsilon_{r2}(\varphi), \quad \upsilon_\varphi = \upsilon_{\varphi 0}(\varphi) + r^2\upsilon_{\varphi 2}(\varphi)$$
$$f = f_0(\varphi) + r^2 f_2(\varphi) \tag{5.1}$$

The functions with the subscript zero correspond to the Prandtl–Mayer solution

$$\upsilon_{r0} = \frac{h-1}{\sqrt{h}}\sin\frac{\varphi}{\sqrt{h}}, \quad \upsilon_{\varphi_0} = -\frac{h-1}{h}\cos\frac{\varphi}{\sqrt{h}}, \quad f_0 = \left(\frac{h-1}{h}\right)^2\cos^2\frac{\varphi}{\sqrt{h}} \tag{5.2}$$

Functions with subscript two are defined by the system

$$\upsilon_{\varphi 0}\upsilon'_{r2} + (2\upsilon_{\varphi 0} + \upsilon'_{r0})\upsilon_{\varphi 2} - 2\upsilon_{r0}\upsilon_{r2} - (h-1)f_2 = 0$$
$$\upsilon_{\varphi 0}\upsilon'_{\varphi 2} + \frac{h-1}{2}f'_2 + (\upsilon'_{\varphi 0} - 3\upsilon_{r0})\upsilon_{\varphi 2} - \upsilon_{\varphi 0}\upsilon_{r2} = 0 \tag{5.3}$$
$$\frac{2}{h-1}\upsilon^2_{\varphi 0}\upsilon'_{\varphi 2} + \upsilon_{\varphi 0}f'_2 + f'_0\upsilon_{\varphi 2} - \frac{6}{h-1}f_0\upsilon_{r2} + \left(-\frac{2h}{h-1}\upsilon_{r0} + \frac{2}{h-1}\upsilon'_{\varphi 0}\right)f_2 = 0$$

From the last two equations we can obtain the final relation for the functions υ_{r_2}, υ_{φ_2}, f_2. Using this, we can reduce system (5.3) to two differential equations, defining the functions υ_{r_2}, υ_{φ_2}, and to the final expression for f_2 in terms of these functions

$$\upsilon_{\varphi 0}\upsilon'_{r2} + \left(\frac{2h}{h+1}\frac{\upsilon^2_{\varphi 0}}{\upsilon_{r0}} - 2\upsilon_{r0}\right)\upsilon_{r2} - \frac{3(h-1)}{h+1}\upsilon_{\varphi 0}\upsilon_{\varphi 2} = 0$$
$$\frac{3h}{h+1}\upsilon_{\varphi 0}\upsilon'_{\varphi 2} - \frac{h}{h+1}\frac{\upsilon^2_{\varphi 0}}{\upsilon_{r0}}\upsilon'_{r2} - \frac{3h}{h+1}\upsilon_{r0}\upsilon_{\varphi 2} - \left(\frac{h+2}{h+1} + \frac{(h-1)^2}{h(h+1)\upsilon^2_{r0}}\right)\upsilon_{\varphi 0}\upsilon_{r2} = 0 \tag{5.4}$$
$$f_2 = \frac{\upsilon_{\varphi 0}}{(h^2-1)\upsilon_{r0}}[(4h-2)\upsilon_{r0}\upsilon_{\varphi 2} - 2h\upsilon_{\varphi 0}\upsilon_{r2}]$$

It follows from the Prandtl–Mayer solution that when $\varphi = \sqrt{(h)}\pi/2$ the function $f_0 = 0$, i.e. the free boundary corresponds to this value of φ when $r \to 0$. Hence, the integration must be carried out within the limits of the variation of φ: $0 \leqslant \varphi \leqslant \sqrt{(h)}\pi/2$. By (3.12) and (3.13) we have the following asymptotic form as $\varphi \to 0$

$$\upsilon_{r2} \approx \frac{3}{2}\varphi + \varphi^2, \quad \upsilon_{\varphi 2} \approx -\frac{1}{2} - \frac{1}{3}\varphi, \quad f_2 \approx -\frac{1}{h}\left(1 + \frac{2}{3}\varphi\right) \tag{5.5}$$

The departure from the origin, according to the asymptotic form (5.5), defines the unique integral curve of system (5.4), the asymptotic form of which as $\varphi \to \sqrt{(h)}\pi/2$ is given by the formulae

$$\upsilon_{r2} \approx \frac{3\sqrt{h}}{h+1}C_1\theta^{1+h}, \quad \upsilon_{\varphi 2} \approx C_1\theta^h$$
$$f_2 \approx -\frac{\theta}{\sqrt{h}(h+1)}\left[\frac{4h-2}{\sqrt{h}}C_1\theta^h + \frac{3\sqrt{h}}{h+1}C_1\theta^{1+h}\right] \tag{5.6}$$

where $\theta = \pi/2 - \varphi/\sqrt{h}$, while C_1 is a constant which depends on h. Hence, as $r \to 0$ in the neighbourhood of the free boundary the asymptotic form of the gas-dynamic functions can be represented in the form

$$\upsilon_r \approx \frac{h-1}{\sqrt{h}}\left(1-\frac{\theta^2}{2}\right)+\frac{3\sqrt{h}}{h+1}C_1\theta^{1+h}r^2+\ldots,\quad \upsilon_\varphi \approx -\frac{h-1}{h}\theta + C_1\theta^h r^2+\ldots$$

$$f \approx \left(\frac{h-1}{h}\right)^2\theta^2 - \frac{C_1\theta^{h+1}}{\sqrt{h}(h+1)}\left[\frac{4h-2}{\sqrt{h}}+\frac{6\sqrt{h}}{h+1}\theta\right]r^2+\ldots \tag{5.7}$$

6. The free boundary is defined by the equations

$$f=0,\quad \frac{dr}{d\theta}=\sqrt{h}r\frac{\upsilon_r - r}{\upsilon_\varphi} \tag{6.1}$$

with initial data $\theta = 0, r = 0$.

In the main terms we have $\upsilon_r \approx (h-1)/\sqrt{h}$, $\upsilon_\varphi \approx -(h-1)\theta/h$ as $\theta \to 0$ and $r \to 0$. Hence

$$dr/d\theta = -hr/\theta \tag{6.2}$$

i.e. according to the initial data when $\theta \to 0$ and $r \to 0$ the equation of the free boundary is $\theta = 0$. When $\theta \to 0$ and for finite values of r, by (5.7), the asymptotic flow in the neighbourhood of the free boundary can be represented in the form

$$\begin{aligned}
\upsilon_r &= \cos\theta\upsilon_{r0}(r)+\theta^{h+1}\upsilon_{r1}(r)+\theta^{h+3}\upsilon_{r2}(r)+\ldots\\
\upsilon_\varphi &= \sin\theta\upsilon_{\varphi 0}(r)+\theta^{h}\upsilon_{\varphi 1}(r)+\theta^{h+2}\upsilon_{\varphi 2}(r)+\ldots\\
f &= \sin^2\theta f_0(r)+\theta^{h+1}f_1(r)+\theta^{h+3}f_2(r)+\ldots
\end{aligned} \tag{6.3}$$

It will be sufficient in what follows to determine the functions with subscripts zero and one. The equations defining the functions with subscript zero, are as follows:

$$\begin{aligned}
&(\upsilon_{r0}-r)\upsilon_{r0}'=0,\quad (\upsilon_{r0}-r)\upsilon_{\varphi 0}'+\frac{1}{\sqrt{hr}}\upsilon_{\varphi 0}^2+\frac{\upsilon_{r0}\upsilon_{\varphi 0}}{r}+\frac{h-1}{\sqrt{hr}}f_0=0\\
&(\upsilon_{r0}-r)f_0'+2\frac{\upsilon_{\varphi 0}}{\sqrt{hr}}f_0+\frac{2}{h-1}\left(\frac{\upsilon_{r0}}{r}+\upsilon_{r0}'+\frac{\upsilon_{\varphi 0}}{\sqrt{hr}}\right)f_0=0
\end{aligned} \tag{6.4}$$

with initial data

$$r=0\to\upsilon_{r0}=\frac{h-1}{\sqrt{h}},\quad \upsilon_{\varphi 0}=-\frac{h-1}{h},\quad f_0=\left(\frac{h-1}{h}\right)^2 \tag{6.5}$$

The constants corresponding to (6.5) will be an obvious solution of this problem.

We will write the system of equations defining the functions with subscript one, taking the values of the functions with subscript zero into account, in the form

$$\begin{aligned}
&\left(\frac{h-1}{\sqrt{h}}-r\right)\frac{d\upsilon_{r1}}{dr}+\frac{h-1}{2}\frac{df_1}{dr}=\frac{h^2-1}{h\sqrt{hr}}\upsilon_{r1}-\frac{(h-1)}{hr}\upsilon_{\varphi 1}\\
&\left(\frac{h-1}{\sqrt{h}}-r\right)\frac{d\upsilon_{\varphi 1}}{dr}=\frac{1}{r}\left(\frac{h-1}{h\sqrt{h}}\upsilon_{\varphi 1}-\frac{h^2-1}{2\sqrt{h}}f_1\right)\\
&\left(\frac{h-1}{\sqrt{h}}-r\right)\frac{df_1}{dr}=\frac{1}{r\sqrt{h}}\left[\frac{-2(h-1)(2h-1)}{h^2}\upsilon_{\varphi 1}+\frac{(h-1)^2}{h}f_1\right]
\end{aligned} \tag{6.6}$$

with initial asymptotic form

$$\upsilon_{\varphi 1} = C_1 r^2, \quad \upsilon_{r1} = \frac{3\sqrt{h}}{h+1} C_1 r^2, \quad f_1 = -\frac{4h-2}{h(h+1)} C_1 r^2 \tag{6.7}$$

The corresponding solution of this system can be written explicitly as follows:

$$\upsilon_{r1} = M^{\frac{h+1}{h}} \left[C_2 + \frac{h-1}{h} C \int M^{3-1/h} r^{-3} \left(3\frac{(h-1)^2}{\sqrt{h}(h+1)} + r \right) dr \right]$$

$$\upsilon_{\varphi_1} = CM^2, \quad f_1 = -\frac{4h-2}{h(h+1)} \upsilon_{\varphi 1}, \quad M = \frac{r}{(h-1)/\sqrt{h} - r} \tag{6.8}$$

It agrees with the initial asymptotic form (5.7) if we put $C = (h-1)^2 C_1/h$ and $C_2 = 0$. Hence, in the neighbourhood of the free boundary when $\theta \to 0$ and $r \to (h-1)/\sqrt{h}$, we have

$$\upsilon_r \approx \frac{h-1}{\sqrt{h}} \cos\theta + 2\frac{(h-1)^3}{h(h+1)} C_1 r^2 \left(\frac{h-1}{\sqrt{h}} - r \right)^{-3} \theta^{h+1} + \ldots$$

$$\upsilon_\varphi \approx -\frac{h-1}{h} \sin\theta + \theta^h C_1 \left[\frac{(h-1)^2}{h} \frac{r^2}{[(h-1)/\sqrt{h} - r]^2} \right] + \ldots \tag{6.9}$$

$$f \approx \left(\frac{h-1}{h} \right)^2 \sin^2\theta + \theta^{h+1} C_1 \left[\frac{(h-1)^2(4h-2)}{h^2(h+1)} \frac{r^2}{[(h-1)/\sqrt{h} - r]^2} \right] + \ldots$$

7. The point $\theta = 0$, $r = (h-1)/\sqrt{h}$ is obviously a singular point for the solution (6.9), and starting from the asymptotic formulae (6.9) the solution in the neighbourhood of this singular point when $h < 5$ can be represented in the form

$$\upsilon_r = \upsilon_{r1}(\psi), \quad \upsilon_\varphi = \theta \upsilon_{\varphi 1}(\psi), \quad f = \theta^2 F_1(\psi)$$

$$\psi = \frac{\sqrt{h}\ |\theta|^{(h-1)/2}}{h-1-\sqrt{h}\, r} \tag{7.1}$$

The corresponding equations for the functions with subscript one then take the following form

$$\upsilon_{r1}' = 0; \quad (2 + \upsilon_{\varphi 1})\upsilon_{\varphi 1}'\psi + \frac{h-1}{2} F_1'\psi = -2\left(\frac{\upsilon_{\varphi 1}^2}{h-1} + \upsilon_{\varphi 1} + F_1 \right)$$

$$(2 + \upsilon_{\varphi 1})F_1'\psi + \frac{2}{h-1} F_1 \upsilon_{\varphi 1}'\psi = -2\left(\frac{2h}{(h-1)^2} F_1 \upsilon_{\varphi 1} + \frac{2}{h-1} F_1 \right) \tag{7.2}$$

while the initial asymptotic forms as $\psi \to 0$ will be

$$\upsilon_{r1} = \frac{h-1}{\sqrt{h}}, \quad \upsilon_{\varphi 1} = -\frac{h-1}{h} + \frac{(h-1)^2}{h} C_1 \psi^2$$

$$F_1 = \left(\frac{h-1}{h} \right)^2 - \frac{(4h-2)(h-1)^2}{h^2(h+1)} C_1 \psi^2 \tag{7.3}$$

By dividing the second equation of system (7.2) by the third, we obtain an equation, the solution of which can be used to reduce the solution of system (7.2) and (7.3) to quadratures. Making the substitution

$$\upsilon_{\varphi 1} = -\frac{h-1}{h} + W, \quad F_1 = \left(\frac{h-1}{h}\right)^2 + \Phi$$

this equation and the initial data for it can be written in the form

$$\frac{d\Phi}{dW} = \frac{[4(2h-1)W + 2(h-1)h(W^2 - \Phi)][h^2\Phi + (h-1)^2]}{(h-1)h[-2(h-1) + (h^3 - 2h^2 + 3h)W + h^2W^2]W + (h-1)h^2(h^2 - 1 - hW)\Phi} \tag{7.4}$$

$$W = 0, \Phi = 0$$

Equation (7.4) has six singular points (W_i, Φ_i) $(i = 1, \ldots, 6)$. For each point the nature of the singularity is determined and, from (7.2), the corresponding value of ψ_i and the asymptotic forms $W_i(\psi)$, $\Phi_i(\psi)$ as $\psi \to \psi_i$.

The singular point $W_1 = 0$, $\Phi_1 = 0$ coincides with the initial data. The nature of the singularity is a saddle, the separatrices of which

$$\text{(1a)} \quad \Phi = \frac{2}{h}W, \quad \text{(1b)} \quad \Phi = -\frac{4h-2}{h(h+1)}W$$

By (7.3) the integral curve of Eq. (7.4) follows separatrice 1b, where the corresponding value is $\psi = \psi_1 = 0$ and the asymptotic forms $\psi \to \psi_1$ is

$$W_1 \approx C\psi^2, \quad \Phi \approx [h(h+1)]^{-1}(4h-2)C\psi^2$$

The singular point $W_2 = -(h+1)/h$, $\Phi_2 = -[(h-1)/h]^2$ is a saddle point, whose separatrices are

$$\text{(2a)} \quad \Phi = \Phi_2, \quad \text{(2b)} \quad \Phi - \Phi_2 \approx 2(h-3)(h-1)^{-1}(W - W_2)$$

The corresponding value of $\psi = \psi_2 (\psi_2 \neq 0)$, and the asymptotic forms of $W(\psi)$ and $\Phi(\psi)$ as $\psi \to \psi_2$ will be in accordance with separatrice 2b

$$W \approx W_2 + \frac{4(h+1)}{h-3}\ln\frac{\psi}{\psi_2}, \quad \Phi \approx \Phi_2 + \frac{8(h+1)}{(h-1)^2}\ln\frac{\psi}{\psi_2}$$

The curve $\psi = \psi_2$ and $\theta \to 0$ is the free boundary, since the corresponding conditions $f(\psi_2) = 0$ are satisfied on it, and as $\theta \to 0$ it coincides with the integral equation $dr/d\theta = \sqrt{h}\, r(\upsilon_r - r)\upsilon_\varphi^{-1}$, or in principal terms $d\psi/d\theta = (h-1)\psi(\upsilon_\varphi + 2)(2\upsilon_\varphi\theta)^{-1}$.

In fact

$$f(\psi_2) = \theta^2 F_1(\psi_2) = \theta^2\{[(h-1)/h]^2 + \Phi_2\} = 0$$

$$\upsilon_{\varphi 1}(\psi_2) = -(h-1)/h + W_2 = -2$$

The singular point $W_3 = (h-1)/h$, $\Phi_3 = -[(h-1)/h]^2$ is a node, the separate whisker of which is $\Phi - \Phi_3 = 2(3-h)(h-1)^{-1}(W - W_3)$, while the curves of the common direction are governed by the asymptotic form

$$W - W_3 \approx C(\Phi - \Phi_3)^{(h-1)/2} + \frac{h-1}{2(3-h)}(\Phi - \Phi_3) \tag{7.5}$$

When $h < 3$ this point corresponds to $|\psi| = \infty$ with asymptotic form $W \approx W_3 + C/\psi$.

The singular point $W_4 = -(h-1)^2/h$, $\Phi_4 = -[(h-1)/h]^2$ is a node, the separate whisker of which is $\Phi - \Phi_4 = [(h-3)/2](W - W_4)$, while the asymptotic form of the curves of the common direction is

$$W - W_4 \approx C(\Phi - \Phi_4)^{1/2} + [2/(h-3)](\Phi - \Phi_4)$$

When $h \leq 3$ the value $\varphi = 0$ corresponds to this point, and $|\psi| = \infty$ when $h > 3$. Here $W - W_4 = \approx C|\psi|^{2/(3-h)}$.

The singular point $W_5 = -2(h-1)/[h(h-2)]$, $\Phi_5 = -4(h^2 - 3h + 1)/[h(h-2)]^2$ is a node, the individual whisker of which $\Phi - \Phi_5 \approx 2(h-3)(h-1)(h-2)^2]^{-1}(W - W_5)$, while the integral curves of the common direction touch the straight line

$$\Phi - \Phi_5 \approx 2(h-3)[(h-1)(h-2)]^{-1}(W - W_5)$$

When the integral curves approach along a common direction, the value $\psi = \psi_5 \neq 0$ corresponds to this point with asymptotic form

$$W - W_5 \approx \frac{2}{h-2}\ln\frac{\psi}{\psi_5}$$

The line $\psi = \psi_5$ when $2 < h < 3$ in the principal order when $\theta \to 0$ and $r \to (h-1)/h$ coincides with the characteristics, which here is defined by the equation

$$\frac{d\psi}{d\theta} = -\frac{(h-1)\psi}{\theta}\left[\frac{\sqrt{h}M \pm \sqrt{F_1[(h-1-\sqrt{h}r)^2 - h\theta^2 N]}}{\left[\sqrt{h}r - (h-1)\right]N} - \frac{1}{2}\right] \tag{7.6}$$

$$M = \upsilon_{\varphi 1}[(h-1)/\sqrt{h} - r], \quad N = F_1 - \upsilon_{\varphi 1}^2$$

Since for finite values of ψ the difference $(h-1)/\sqrt{h} - r \simeq \theta^{(h-1)/2}$, when $h < 3$ Eq. (7.6) asymptotically acquires the form

$$\frac{d\psi}{d\theta} \approx \frac{h-1}{2}\frac{\psi}{\theta}\left(1 - \frac{2}{\sqrt{F} \pm \upsilon_{\varphi 1}}\right)$$

By virtue of the asymptotic forms derived earlier, this equation can be rewritten in the form $d\psi/d\theta \approx h(\psi - \psi_5)/[(h-2)\theta]$, i.e. $\psi = \psi_5 + A\theta^{h/(h-2)}$, where A is a constant which depends on h.

The singular point $|W_6| = \infty$, $\Phi_6 = \infty$ is a node. The asymptotic form of the curves of the common direction has the form $\Phi \approx CW^2$, and the corresponding value $\psi = \psi_6 = -0$ for the asymptotic form $W \approx A|\psi|^{2/(h-1)}$, $\Phi \approx CA^2|\psi|^{4/(h-1)}$. Here $\upsilon_\varphi \simeq (h-1)/\sqrt{h} - r, f \simeq [(h-1)/\sqrt{h} - r]^2$ and the line $\psi =$

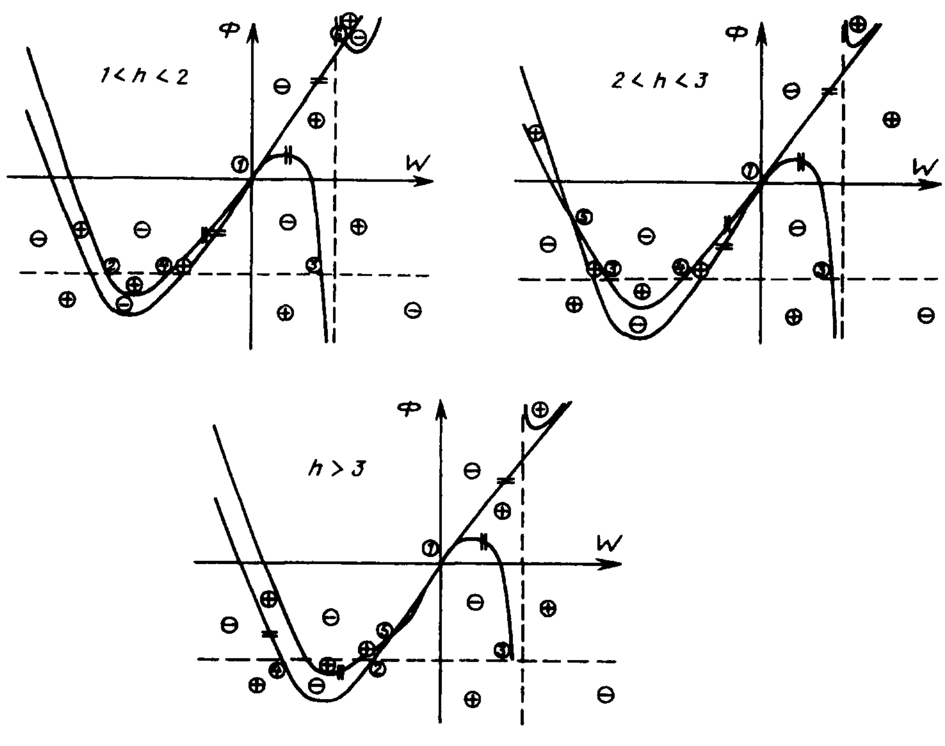

Fig. 1.

0 corresponds to the characteristic. This follows from the asymptotic form of the equation of the characteristics in this case: $d\psi/d\theta \approx (h-1)\psi/(2\theta)$, i.e. $\psi \approx B|\theta|^{(h-1)/2}$, where B is a constant which depends on h.

From the scheme of isoclines (see Fig. 1) and the values of the variable ψ derived above at the corresponding singular points one can determine the form of the integral curve (the plus and minus signs in the figure correspond to the sign of the derivative $d\Phi/dW$ in the region bounded by the zero and infinity isoclines; the horizontal dashed lines correspond to $A = -[(h-1)/h]^2$ and the vertical dashed lines correspond to the value $W = (h^2-1)/h$). Here one must take into account the fact that the integral curve, in the final analysis, must intersect the point corresponding to the free boundary, i.e. the singular point W_2, Φ_2.

When $h < 3$ the required integral curve, emerging from the first singular point, corresponding to $\psi = +0$, passes through the point W_3, Φ_3, corresponding to the value $\psi = \pm\infty$, touches the line $\Phi = -[(h-1)/h]^2$, and after that it proceeds to the infinitely distant point $W_6 = \pm\infty$, $\Phi_6 = \infty$. Its subsequent motion depends on whether $h < 2$ or $h > 2$.

Scheme 1: $1 < h < 2$. At an infinitely distant point, by varying the direction, which is possible since the value of $\psi = -0$ corresponding to it in the principal term coincides with the equation of the characteristic, it passes through the saddle point W_2, Φ_2 along its separatrice.

Scheme 2: $2 < h < 3$. Taking into account the fact that the integral curve, as $W = -\infty$, $\Phi = \infty$, passes to the left of the zero and infinity isoclines of Eq. (7.4), we conclude that when it is extended to $W > -\infty$ it intersects the nodal point W_5, Φ_5 in the common direction, after which, changing direction, it arrives at the saddle point W_2, Φ_2. A change of direction is possible since the point W_5, Φ_5 corresponds to the line $\psi = \psi_3$, the equation of which agrees in its principal terms with the equation of the characteristic.

Hence, when $h < 3$ the motion of the integral curve is completed on the line $\psi = \psi_2$.

Scheme 3: $h > 3$. In this case, in the neighbourhood of the singular point W_3, Φ_3 solution (7.5) in the principal terms can be represented in the form

$$W = \frac{h-1}{2(3-h)}\Phi \tag{7.7}$$

i.e. the extension of the integral curve of Eq. (7.4) when $W > W_3$ would lead to negative values of $F(\psi)$, and consequently, of $f(\psi)$, which is contradictory. Hence, when $h > 3$ the integral curve of Eq. (7.6) emerges along the separatrice 1b in the direction $\Phi > 0$, corresponding to the initial values of the singular point W_1, Φ, i.e. towards negative values of the variable ψ. Then the separatrice, according to the scheme of the isoclines 3, intersects the zero isocline, and then the infinity isocline, and as a result intersects the node corresponding to the singular point W_4, Φ_4, along the common direction. At this point $\upsilon_{\varphi 1} = -(h-1)$ and $F_1 = 0$, where $d\Phi/dW = 0$, and the corresponding value is $\psi = -\infty$, $\upsilon_{\varphi 1} \approx |\psi|^{2/(3-h)}$, $F_1 \approx |\psi|^{4/(3-h)}$. Consequently, the derivatives of all orders in ψ of the functions $\upsilon_{\varphi 1}$ and F_1 at this point vanish, by virtue of which further motion of the integral curve is permissible along the line $F_1 = [(h-1)/h]^2 + \Phi = 0$, which is the integral of Eq. (7.4), up to the singular point W_2, Φ_2. Hence, the equation of the free boundary for all values of $1 < h < 5$ ($3/2 < \gamma < \infty$) in the neighbourhood of the point ($\theta = 0$, $r = (h-1)/\sqrt{h}$ can be represented in the form

$$r \approx \frac{h-1}{\sqrt{h}} - \frac{(-\theta)^{(h-1)/2}}{\psi_2} \tag{7.8}$$

i.e. it deviates from the line $\theta = 0$.

Note that the line $\psi = -\infty$ also satisfies the conditions of the free boundary. However, the possibility of a further advance of the integral curve to the singular point W_2, Φ_2 and the continuity of the expected results with respect to the parameter h, enable us to assume that the free boundary is defined by Eq. (7.8) when $h > 3$.

This investigation was suggested by A. N. Kraiko. We also discussed our preliminary results with him, which led to some corrections. The manuscript was prepared by M. S. Gavreyeva and E. A. Zabrodina. The graphs were drawn by A. V. Severin. I express my grateful thanks to all of them.

REFERENCES

1. STANYUKOVICH K. R., *Unsteady Motions of a Continuous Medium*. Gostekhizdat, Moscow, 1955.
2. SUCHKOV V. A., Flow into a vacuum along a sloping wall. *Prikl. Mat. Mekh.* **27**, 4, 739–740, 1963.
3. BAUTIN S. P. and DERYABIN S. L., The escape of an ideal gas into a vacuum. *Dokl. Akad. Nauk SSSR* **273**, 4, 817–820, 1983.

4. KRAIKO A. N., Asymptotic features of the unsteady expansion of an ideal gas into a vacuum. *Prikl. Mekh.* **58,** 4, 70–80, 1994.
5. VAN DYKE M., *Perturbation Methods in Fluid Mechanics*. Mir, Moscow, 1967.
6. KAZHDAN Ya. M., The motion of a gas behind a diverging detonation wave in a space with a truncated cone. *Prikl. Mat. Mekh.* **31,** 5, 916–928, 1967.
7. IL'IN A. M., *Matching of Asymptotic Expansions of the Solutions of Boundary-value Problems*. Nauka, Moscow, 1989.

Translated by R.C.G.